

Almost Birkhoff Theorem in General Relativity

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We extend Birkhoff's theorem for almost LRS-II vacuum spacetimes to show that the rigidity of spherical vacuum solutions of Einstein's field equations continues even in the perturbed scenario.

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I. INTRODUCTION

The core content of Birkhoff's theorem [1] is that any spherically symmetric solution of the vacuum field equations has an extra symmetry: it must be either locally static or spatially homogeneous. This underlies the crucial importance in astrophysics of the Schwarzschild solution, as it means that the exterior metric of any exactly spherical star must be given by the Schwarzschild metric (it cannot be Minkowski spacetime if the mass is non-zero); and this also underlies the uniqueness results for non-rotating black holes.

However it is an exact theorem that is only valid for exact spherical symmetry; but no real star is exactly spherically symmetric. So a key question is whether the result is approximately true for approximately spherically symmetric vacuum solutions. We prove an “almost Birkhoff theorem” in this paper that shows this is indeed the case, so those results carry over to astrophysically realistic situations (such as the Solar System). There are of course many papers discussing perturbations of the Schwarzschild solution, but none appear to focus on this specific issue. It is in a sense an analogue of another important result, the “almost EGS theorem” [2] that generalizes from exact isotropy of radiation to approximate isotropy the crucial Ehlers-Geren-Sachs theorem [3], proving that isotropic cosmic background radiation everywhere in an expanding universe domain U implies a Robertson-Walker geometry in that domain. In both cases an exact mathematical result, depending on exact symmetry, is generalized to a more physically realistic result, depending on approximate symmetry.

The almost-Birkhoff result will not be true if spacetime is not a vacuum (empty) spacetime, for the degrees of freedom available through a matter source generically invalidate the result, as is shown for example by the family of Lemaître-Tolman-Bondi (LTB) models [4]. However it remains true for the electrovac case [5] and solutions with a cosmological constant. It will also not be true for gravitational theories with a scalar degree of freedom, such as scalar-tensor theories [6]. The rigidity embodied in this property of the Einstein Field Equations is specific to

vacuum General Relativity solutions, or solutions with a trace-free matter tensor. One should also note that (like the EGS result) the result is local: both Birkhoff's theorem, and our generalization of it, are independent of boundary conditions at infinity: they hold in local neighborhoods U . There are of course crucial differences between the exactly spherical case and that of approximate symmetry: specifically, an almost spherically symmetric pulsating star can emit gravitational waves, but this is not possible in the exactly spherical case. However we show that in such a case if the intensity of the radiation is such as to leave the solution almost spherically symmetric, the space time will remain almost static, which is defined in a way we will make precise later.

We prove the result by using the 1+1+2 covariant perturbation formalism [7, 8], which developed from the 1+3 covariant perturbation formalism [9]. This enable us to prove the approximate result as a straightforward generalization of the exact result, which we prove first, using the 1+1+2 formalism. Actually we prove a small generalization of the standard Birkhoff result: it holds for all Class II Locally Rotational Symmetric (LRS) spacetimes [10] (which include Schwarzschild as a special case).

II. 1+1+2 COVARIANT SPLITTING OF SPACETIME

The 1+3 covariant formalism [9] has proven to be a very useful technique in many aspects of relativistic cosmology. In this approach first we define a timelike congruence by a timelike unit vector u^a ($u^a u_a = -1$). Then the spacetime is split in the form $R \otimes V$ where R denotes the timeline along u^a and V is the tangent 3-space perpendicular to u^a . Then any vector X^a can be projected on the 3-space by the projection tensor $h_b^a = g_b^a + u^a u_b$. The vector u^a is used to define the *covariant time derivative* (denoted by a dot) for any tensor $T^{a..b}{}_{c..d}$ along the observers' worldlines defined by

$$\dot{T}^{a..b}{}_{c..d} = u^e \nabla_e T^{a..b}{}_{c..d}, \quad (1)$$

and the tensor h_{ab} is used to define the fully orthogonally projected *covariant derivative* D for any tensor $T^{a..b}{}_{c..d}$,

$$D_e T^{a..b}{}_{c..d} = h^a{}_f h^p{}_c ... h^b{}_g h^q{}_d h^r{}_e \nabla_r T^{f..g}{}_{p..q}, \quad (2)$$

with total projection on all the free indices.

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In the (1+1+2) approach we further split the 3-space V , by introducing the spacelike unit vector e^a orthogonal to u^a so that

$$e_a u^a = 0, \quad e_a e^a = 1. \quad (3)$$

Then the *projection tensor*

$$N_a{}^b \equiv h_a{}^b - e_a e^b = g_a{}^b + u_a u^b - e_a e^b, \quad N^a{}_a = 2, \quad (4)$$

projects vectors onto the tangent 2-surfaces orthogonal to e^a and u^a , which, following [7], we will refer to as ‘sheets’. Hence it is obvious that $e^a N_{ab} = 0 = u^a N_{ab}$. In (1+3) approach any second rank symmetric 4-tensor can be split into a scalar along u^a , a 3-vector and a *projected symmetric trace free* (PSTF) 3-tensor. In (1+1+2) slicing, we can take this split further by splitting the 3-vector and PSTF 3-tensor with respect to e^a . Any 3-vector, ψ^a , can be irreducibly split into a component along e^a and a sheet component Ψ^a , orthogonal to e^a i.e.

$$\psi^a = \Psi e^a + \Psi^a, \quad \Psi \equiv \psi^a e_a, \quad \Psi^a \equiv N^{ab} \psi_b. \quad (5)$$

A similar decomposition can be done for PSTF 3-tensor, ψ_{ab} , which can be split into scalar (along e^a), 2-vector and 2-tensor part as follows:

$$\psi_{ab} = \psi_{\langle ab \rangle} = \Psi \left(e_a e_b - \frac{1}{2} N_{ab} \right) + 2\Psi_{(a} e_{b)} + \Psi_{ab}, \quad (6)$$

where

$$\begin{aligned} \Psi &\equiv e^a e^b \psi_{ab} = -N^{ab} \psi_{ab}, \\ \Psi_a &\equiv N_a{}^b e^c \psi_{bc}, \\ \Psi_{ab} &\equiv \psi_{\{ab\}} \equiv \left(N^c {}_{(a} N_{b)}{}^d - \frac{1}{2} N_{ab} N^{cd} \right) \psi_{cd}, \end{aligned} \quad (7)$$

and the curly brackets denote the PSTF part of a tensor with respect to e^a .

We also have

$$h_{\{ab\}} = 0, \quad N_{\langle ab \rangle} = -e_{\langle a} e_{b\rangle} = N_{ab} - \frac{2}{3} h_{ab}. \quad (8)$$

The sheet carries a natural 2-volume element, the alternating Levi-Civita 2-tensor:

$$\varepsilon_{ab} \equiv \varepsilon_{abc} e^c = \eta_{abcd} e^c u^d, \quad (9)$$

where ε_{abc} is the 3-space permutation symbol the volume element of the 3-space and η_{abcd} is the space-time permutator or the 4-volume. With these definitions it follows that any 1+3 quantity can be locally split in the 1+1+2 setting into only three types of objects: scalars, 2-vectors in the sheet, and PSTF 2-tensors (also defined on the sheet).

Now apart from the ‘time’ (dot) derivative of an object (scalar, vector or tensor) which is the derivative along the timelike congruence u^a , we now introduce two new derivatives, which e^a defines, for any object $\psi_{a...b}{}^{c...d}$:

$$\hat{\psi}_{a...b}{}^{c...d} \equiv e^f D_f \psi_{a...b}{}^{c...d}, \quad (10)$$

$$\delta_f \psi_{a...b}{}^{c...d} \equiv N_a{}^f ... N_b{}^g N_h{}^c ... N_i{}^d N_f{}^j D_j \psi_{f...g}{}^{i...j}. \quad (11)$$

The hat-derivative is the derivative along the e^a vector-field in the surfaces orthogonal to u^a . The δ -derivative is the projected derivative onto the orthogonal 2-sheet, with the projection on every free index. We can now decompose the covariant derivative of e^a in the direction orthogonal to u^a into its irreducible parts giving

$$D_a e_b = e_a a_b + \frac{1}{2} \phi N_{ab} + \xi \varepsilon_{ab} + \zeta_{ab}, \quad (12)$$

where

$$a_a \equiv e^c D_c e_a = \hat{e}_a, \quad (13)$$

$$\phi \equiv \delta_a e^a, \quad (14)$$

$$\xi \equiv \frac{1}{2} \varepsilon^{ab} \delta_a e_b, \quad (15)$$

$$\zeta_{ab} \equiv \delta_{\{a} e_{b\}}. \quad (16)$$

We see that along the spatial direction e_a , ϕ represents the *expansion of the sheet*, ζ_{ab} is the *shear of e^a* (i.e. the distortion of the sheet) and a^a its *acceleration*. We can also interpret ξ as the *vorticity* associated with e^a so that it is a representation of the “twisting” or rotation of the sheet. The other derivative of e^a is its change along u^a ,

$$\dot{e}_a = \mathcal{A} u_a + \alpha_a, \quad (17)$$

where we have $\mathcal{A} = e^a \dot{u}_a$ and $\alpha_a = N_{ac} \dot{e}^c$. Also we can write the (1+3) kinematical variables and Weyl tensor as follows

$$\Theta = h_a^b \nabla_b u^a \quad (18)$$

$$\dot{u}^a = \mathcal{A} e^a + \mathcal{A}^a, \quad (19)$$

$$\omega^a = \Omega e^a + \Omega^a, \quad (20)$$

$$\sigma_{ab} = \Sigma \left(e_a e_b - \frac{1}{2} N_{ab} \right) + 2\Sigma_{(a} e_{b)} + \Sigma_{ab}, \quad (21)$$

$$E_{ab} = \mathcal{E} \left(e_a e_b - \frac{1}{2} N_{ab} \right) + 2\mathcal{E}_{(a} e_{b)} + \mathcal{E}_{ab}, \quad (22)$$

$$H_{ab} = \mathcal{H} \left(e_a e_b - \frac{1}{2} N_{ab} \right) + 2\mathcal{H}_{(a} e_{b)} + \mathcal{H}_{ab}. \quad (23)$$

where E_{ab} and H_{ab} are the electric and magnetic part of the Weyl tensor respectively. Therefore the key variables of the 1+1+2 formalism are

$$\begin{aligned} &[\Theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \phi, \xi, \mathcal{A}^a, \Omega^a, \Sigma^a, \alpha^a, a^a, \mathcal{E}^a, \mathcal{H}^a, \\ &\Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}, \zeta_{ab}] \end{aligned} \quad (24)$$

These variables (scalars, 2-vectors and PSTF 2-tensors) form an *irreducible set* and completely describe a vacuum spacetime. In terms of these variables the full covariant derivatives of e^a and u^a are

$$\begin{aligned} \nabla_a e_b &= -\mathcal{A} u_a u_b - u_a \alpha_b + \left(\Sigma + \frac{1}{3} \Theta \right) e_a u_b \\ &+ (\Sigma_a - \varepsilon_{ac} \Omega^c) u_b + e_a a_b + \frac{1}{2} \phi N_{ab} \\ &+ \xi \varepsilon_{ab} + \zeta_{ab}, \end{aligned} \quad (25)$$

$$\begin{aligned}\nabla_a u_b &= -u_a (\mathcal{A} e_b + \mathcal{A}_b) + e_a e_b \left(\frac{1}{3} \Theta + \Sigma \right) \\ &\quad + e_a (\Sigma_b + \varepsilon_{bc} \Omega^c) + (\Sigma_a - \varepsilon_{ac} \Omega^c) e_b \\ &\quad + N_{ab} \left(\frac{1}{3} \Theta - \frac{1}{2} \Sigma \right) + \Omega \varepsilon_{ab} + \Sigma_{ab}.\end{aligned}\quad (26)$$

For the complete set of evolution equations, propagation equations, mixed equations and constraints for the above irreducible set of variables please see equations (48-81) of [8]. Also we have the following commutation relations for the different derivatives of any scalar ψ ,

$$\hat{\psi} - \dot{\psi} = -\mathcal{A}\dot{\psi} + \left(\frac{1}{3} \Theta + \Sigma \right) \hat{\psi} + (\Sigma_a + \varepsilon_{ac} \Omega^c - \alpha_a) \delta^a \psi, \quad (27)$$

$$\delta_a \delta_b \psi - \delta_b \delta_a \psi = 2\varepsilon_{ab} (\Omega \dot{\psi} - \xi \hat{\psi}) + 2a_{[a} \delta_{b]} \psi \quad (28)$$

From the above two relations it is clear that the 2-sheet is a genuine two surface (rather than just a collection of tangent planes), in the sense that the commutator of the time and hat derivative do not depend on any sheet component and also the sheet derivatives commute, if and only if $\Sigma_a + \varepsilon_{ac} \Omega^c - \alpha_a = 0$ and $\Omega = \xi = a^a = 0$.

III. BIRKHOFF THEOREM FOR VACUUM LRS-II SPACETIMES

Locally Rotationally Symmetric (LRS) spacetimes possess a continuous isotropy group at each point and hence a multi-transitive isometry group acting on the spacetime manifold [10]. These spacetimes exhibit locally (at each point) a unique preferred spatial direction, covariantly defined. Since LRS spacetimes are defined to be isotropic about a preferred direction, this allows for the vanishing of *all* orthogonal 1+1+2 vectors and tensors, such that there are no preferred directions in the sheet. Then, all the non-zero 1+1+2 variables are covariantly defined scalars. A subclass of the LRS spacetimes, called LRS-II, contains all the LRS spacetimes that are rotation free. As consequence in LRS-II spacetimes the variables Ω , ξ and \mathcal{H} are identically zero and the variables $\{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}\}$ fully characterize the kinematics of the vacuum spacetime. The propagation and evolution equations of these variables are:

$$\hat{\phi} = -\frac{1}{2} \phi^2 + \left(\frac{1}{3} \Theta + \Sigma \right) \left(\frac{2}{3} \Theta - \Sigma \right) - \mathcal{E} \quad (29)$$

$$\hat{\Sigma} - \frac{2}{3} \hat{\Theta} = -\frac{3}{2} \phi \Sigma, \quad (30)$$

$$\hat{\mathcal{E}} = -\frac{3}{2} \phi \mathcal{E}. \quad (31)$$

$$\dot{\phi} = -\left(\Sigma - \frac{2}{3} \Theta \right) \left(\mathcal{A} - \frac{1}{2} \phi \right), \quad (32)$$

$$\dot{\Sigma} - \frac{2}{3} \dot{\Theta} = -\mathcal{A} \phi + 2 \left(\frac{1}{3} \Theta - \frac{1}{2} \Sigma \right)^2 - \mathcal{E}, \quad (33)$$

$$\dot{\mathcal{E}} = \left(\frac{3}{2} \Sigma - \Theta \right) \mathcal{E}. \quad (34)$$

$$\hat{\mathcal{A}} - \dot{\Theta} = -(\mathcal{A} + \phi) \mathcal{A} + \frac{1}{3} \Theta^2 + \frac{3}{2} \Sigma^2. \quad (35)$$

Since the vorticity vanishes, the unit vector field u^a is hypersurface-orthogonal to the spacelike 3-surfaces whose intrinsic curvature can be calculated from the *Gauss equation* for u^a that is generally given as [11]:

$${}^{(3)}R_{abcd} = (R_{abcd})_\perp - K_{ac} K_{bd} + K_{bc} K_{ad}, \quad (36)$$

where ${}^{(3)}R_{abcd}$ is the *3-curvature tensor*, \perp means projection with h_{ab} on all indices and K_{ab} is the *extrinsic curvature*. Also we note that in case of LRS-II spacetimes the sheets at each point mesh together to form 2-surfaces. The Gauss equation for e^a together with the 3-Ricci identities determine the 3-Ricci curvature tensor of the spacelike 3-surfaces orthogonal to u^a to be

$${}^3R_{ab} = -\left[\hat{\phi} + \frac{1}{2} \phi^2 \right] e_a e_b - \left[\frac{1}{2} \hat{\phi} + \frac{1}{2} \phi^2 - K \right] N_{ab}. \quad (37)$$

This gives the 3-Ricci-scalar as

$${}^3R = -2 \left[\frac{1}{2} \hat{\phi} + \frac{3}{4} \phi^2 - K \right] \quad (38)$$

where K is the *Gaussian curvature* of the sheet, ${}^2R_{ab} = K N_{ab}$. From this equation and (29) an expression for K is obtained in the form [11]

$$K = -\mathcal{E} + \frac{1}{4} \phi^2 - \left(\frac{1}{3} \Theta - \frac{1}{2} \Sigma \right)^2 \quad (39)$$

From (29-34), the evolution and propagation equations of K can be determined as

$$\dot{K} = -\frac{2}{3} \left(\frac{2}{3} \Theta - \Sigma \right) K, \quad (40)$$

$$\hat{K} = -\phi K. \quad (41)$$

From equations (31), (34), (40) and (41) we get an interesting geometrical result for vacuum LRS-II spacetime

$$\mathcal{E} = C K^{3/2}. \quad (42)$$

That is, the 1+1+2 scalar of the electric part of the Weyl tensor is always proportional to a power of the Gaussian curvature of the 2-sheet. The proportionality constant C sets up a scale in the problem. We can immediately see that for Minkowski spacetime $C = 0$.

To covariantly investigate the geometry of the vacuum LRS-II spacetime, let us try to solve the *Killing equation* for a Killing vector of the form $\xi_a = \Psi u_a + \Phi e_a$, where Ψ and Φ are scalars. The Killing equation gives

$$\nabla_a (\Psi u_b + \Phi e_b) + \nabla_b (\Psi u_a + \Phi e_a) = 0. \quad (43)$$

Using equations (25) and (26), and multiplying the Killing equation by $u^a u^b$, $u^a e^b$, $e^a e^b$ and N^{ab} we get the

following differential equations and constraints:

$$\dot{\Psi} + \mathcal{A}\Phi = 0, \quad (44)$$

$$\dot{\Psi} - \dot{\Phi} - \Psi\mathcal{A} + \Phi(\Sigma + \frac{1}{3}\Theta) = 0; \quad (45)$$

$$\hat{\Phi} + \Psi(\frac{1}{3}\Theta + \Sigma) = 0, \quad (46)$$

$$\Psi(\frac{2}{3}\Theta - \Sigma) + \Phi\phi = 0. \quad (47)$$

Now we know $\xi_a\xi^a = -\Psi^2 + \Phi^2$. If ξ^a is timelike (that is $\xi_a\xi^a < 0$), then because of the arbitrariness in choosing the vector u^a , we can always make $\Phi = 0$. On the other hand, if ξ^a is spacelike (that is $\xi_a\xi^a > 0$), we can make $\Psi = 0$.

Let us first assume that ξ^a is timelike and $\Phi = 0$. In that case we know that the solution of equations (44) and (45) always exists while the constraints (46) and (47) together imply that in general, (for a non trivial Ψ), $\Theta = \Sigma = 0$. Thus the expansion and shear of an unit vector field along the timelike Killing vector vanishes. In this case the spacetime is static. Now if ξ^a is spacelike and $\Psi = 0$, solution of equations (45) and (46) always exists and the constraints (44) and (47) together imply that in general, (for a non trivial Φ), $\phi = \mathcal{A} = 0$. From the LRS-II equations (29)- (35), we then immediately see that the spatial derivatives of all quantity vanish and hence the spacetime is spatially homogeneous.

In other words, we can say: *There always exists a Killing vector in the local $[u, e]$ plane for a vacuum LRS-II spacetime. If the Killing vector is timelike then the spacetime is locally static, and if the Killing vector is spacelike the spacetime is locally spatially homogeneous.*

In fact in the first case, when $\Theta = \Sigma = 0$, we have $\dot{K} = 0$. Furthermore if choose coordinates to make the Gaussian curvature ‘ K ’ of the spherical sheets proportional to the inverse square of the radius co-ordinate ‘ r ’, (such that this coordinate becomes the *area radius* of the sheets), then this geometrically relates the ‘hat’ derivative with the radial co-ordinate ‘ r ’. As we have already seen, $\hat{K} = -\phi K$, where the hat derivative, defined in terms of the derivative with respect to the co-ordinate ‘ r ’, depends on the specific choice of e^a (orthogonal to u^a and the sheet). If we choose the ‘radial’ co-ordinate as the area radius of the spherical sheets, then from (10) and (14) the hat derivative of any scalar M becomes

$$\hat{M} = \frac{1}{2}r\phi\frac{dM}{dr}. \quad (48)$$

for a static spacetime. If the spacetime is not static then there is also a dot derivative in the RHS of equation (48). Details are given in [11].

Now solving equations (29)- (35), we get the unique

solution

$$\phi = \frac{2}{r}\sqrt{1 - \frac{2m}{r}}, \quad \mathcal{A} = \frac{m}{r^2}\left[1 - \frac{2m}{r}\right]^{-\frac{1}{2}} \quad (49)$$

$$\mathcal{E} = \frac{2m}{r^3}, \quad K = \frac{1}{r^2} \quad (50)$$

Here the constant m is the constant of integration. Solving for the metric components using the definition of these geometrical quantities we get [11]

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{(1 - \frac{2m}{r})} + r^2d\Omega^2, \quad (51)$$

which is the metric of a static Schwarzschild exterior. In the second case, when $\phi = \mathcal{A} = 0$, we can choose $u^a = \sqrt{\frac{2m}{t} - 1}\delta_t^a$ where m is a constant and then solving (29)- (35), we get the unique solution

$$\Theta = \frac{3m - 2t}{t\sqrt{t(2m - t)}}, \quad \Sigma = -\frac{2}{3}\frac{3m - t}{t\sqrt{t(2m - t)}}, \quad (52)$$

$$\mathcal{E} = -\frac{2m}{t^3}, \quad K = \frac{1}{t^2} \quad (53)$$

Again solving for the metric components we get

$$ds^2 = -\frac{dt^2}{(\frac{2m}{t} - 1)} + \left(\frac{2m}{t} - 1\right)dr^2 + t^2d\Omega^2, \quad (54)$$

which is a part of the Schwarzschild solution inside the Schwarzschild radius.

Thus we have proved the (local) **Birkhoff Theorem**: *Any C^2 solution of Einstein’s equations in empty space, which is of the class LRS-II in an open set \mathcal{S} , is locally equivalent to part of maximally extended Schwarzschild solution is \mathcal{S} .* Also it is interesting to note that the modulus of the proportionality constant in equation (42), which sets a scale in the problem, is exactly equal to the Schwarzschild radius.

IV. ALMOST BIRKHOFF THEOREM

As we have already seen, for a spherically symmetric spacetime, the 1+1+2 scalar of the electric part of the Weyl tensor is always proportional to the $(3/2)^{th}$ power of the Gaussian curvature of the 2-sheet, with the proportionality constant defining a scale in the problem. Let us now perturb this geometry and define the notion of an *almost spherically symmetric* spacetime in the following way:

Any C^2 spacetime that admits a local 1+1+2 splitting at every point such that the magnitude of all 2-vectors on the sheet, the sheet gradient of scalars (defined by $\sqrt{\psi_a\psi^a}$), the magnitude of all PSTF 2-tensors on the sheet, and the sheet derivative of 2-vectors (defined by

$\sqrt{\psi_{ab}\psi^{ab}}$) at any given point are either zero or much smaller than the scale defined by the modulus of the proportionality constant in equation (42), is called an almost spherically symmetric spacetime.

We would like to emphasize here that though Minkowski spacetime belongs to the set of LRS-II, in the above definition of the perturbed spacetime we exclude the Minkowski background, as in that case the scale is identically zero. As we have seen from equations (27) and (28), the sheet will be a genuine two surface if and only if the commutator of the time and hat derivative do not depend on any sheet component and also the sheet derivatives commute. In the perturbed scenario we will require the sheet to be an almost genuine 2-surface in the sense that the commutator of the time and hat derivative almost do not depend on any sheet component and the sheet derivatives almost commute; this will follow from our definition of almost spherically symmetric. In that case the scalars Ω and ξ would be of the same order of smallness as the other vectors and PSTF 2-tensors on the sheet. Also using the constraint equation

$$\delta_a \Omega^a + \varepsilon_{ab} \delta^a \Sigma^b = (2\mathcal{A} - \phi) \Omega - 3\xi \Sigma + \varepsilon_{ab} \zeta^{ac} \Sigma_c^b + \mathcal{H}, \quad (55)$$

we see the scalar \mathcal{H} also is of the same order of smallness. Hence the set of 1+1+2 variables

$$[\Omega, \mathcal{H}, \xi, \mathcal{A}^a, \Omega^a, \Sigma^a, \alpha^a, a^a, \mathcal{E}^a, \mathcal{H}^a, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}, \zeta_{ab}] . \quad (56)$$

are all of $\mathcal{O}(\epsilon)$ with respect to the invariant scale. Using equations (48-81) of [8], we can get the propagation and evolution equations of these small quantities. We now list all the equations up to the first order (that is up to order ' ϵ '). The time evolution equations of ξ and $\zeta_{\{ab\}}$ are as follows:

$$\dot{\xi} = \left(\frac{1}{2}\Sigma - \frac{1}{3}\theta\right)\xi + (\mathcal{A} - \frac{1}{2}\phi)\Omega + \frac{1}{2}\varepsilon_{ab}\delta^a\alpha^b + \frac{1}{2}\mathcal{H}, \quad (57)$$

$$\dot{\zeta}_{\{ab\}} = \left(\frac{1}{2}\Sigma - \frac{1}{3}\theta\right)\zeta_{ab} + (\mathcal{A} - \frac{1}{2}\phi)\Sigma_{ab} + \delta_{\{a}\alpha_{b\}} - \varepsilon_{c\{a}\mathcal{H}_{b\}}^c. \quad (58)$$

The Vorticity evolution equations:

$$\dot{\Omega} = \frac{1}{2}\varepsilon_{ab}\delta^a\mathcal{A}^b + \mathcal{A}\xi + \Omega\left(\Sigma - \frac{2}{3}\theta\right), \quad (59)$$

$$\dot{\Omega}_a + \frac{1}{2}\varepsilon_{ab}\hat{\mathcal{A}}^b = -\left(\frac{2}{3}\theta + \frac{1}{2}\Sigma\right)\Omega_a + \frac{1}{2}\varepsilon_{ab}\left[-\mathcal{A}a^b + \delta^b\mathcal{A} - \frac{1}{2}\phi\mathcal{A}^b\right]. \quad (60)$$

Shear evolution equations:

$$\dot{\Sigma}_{\{ab\}} = \delta_{\{a}\mathcal{A}_{b\}} + \mathcal{A}\zeta_{ab} - \left(\frac{2}{3}\theta + \frac{1}{2}\Sigma\right)\Sigma_{ab} - \mathcal{E}_{ab}, \quad (61)$$

$$\dot{\Sigma}_a - \frac{1}{2}\hat{\mathcal{A}}_a = \frac{1}{2}\delta_a\mathcal{A} + (\mathcal{A} - \frac{1}{4}\phi)\mathcal{A}_a - \left(\frac{2}{3}\theta + \frac{1}{2}\Sigma\right)\Sigma_a + \frac{1}{2}\mathcal{A}a_a - \frac{3}{2}\Sigma\alpha_a - \mathcal{E}_a \quad (62)$$

Evolution equation for \hat{e}_a :

$$\hat{\alpha}_a - \dot{a}_a = -\left(\frac{1}{2}\phi + \mathcal{A}\right)\alpha_a + \left(\frac{1}{3}\theta + \Sigma\right)(\mathcal{A}_a - a_a) + \left(\frac{1}{2}\phi - \mathcal{A}\right)(\Sigma_a + \varepsilon_{ab}\Omega^b) - \varepsilon_{ab}\mathcal{H}^b. \quad (63)$$

Electric Weyl evolution:

$$\begin{aligned} \dot{\mathcal{E}}_a + 12\varepsilon_{ab}\hat{\mathcal{H}}^b &= \frac{3}{4}\varepsilon_{ab}\delta^b\mathcal{H} + \frac{1}{2}\varepsilon_{bc}\delta^b\mathcal{H}_a^c - \frac{3}{4}\mathcal{E}\Sigma_a \\ &+ \frac{3}{4}\mathcal{E}\varepsilon_{ab}\Omega^b - \frac{3}{2}\mathcal{E}\alpha_a + \left(\frac{3}{4}\Sigma - \theta\right)\mathcal{E}_a \\ &- \left(\frac{1}{4}\phi + \mathcal{A}\right)\varepsilon_{ab}\mathcal{H}^b, \end{aligned} \quad (64)$$

$$\begin{aligned} \dot{\mathcal{E}}_{\{ab\}} - \varepsilon_{c\{a}\hat{\mathcal{H}}_{b\}}^c &= -\varepsilon_{c\{a}\delta^c\mathcal{H}_{b\}} - \frac{3}{2}\mathcal{E}\Sigma_{ab} - \left(\theta + \frac{3}{2}\Sigma\right)\mathcal{E}_{ab} \\ &+ \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{c\{a}\mathcal{H}_{b\}}^c. \end{aligned} \quad (65)$$

Magnetic Weyl evolution:

$$\dot{\mathcal{H}} = -\varepsilon_{ab}\delta^a\mathcal{E}^b - 3\xi\mathcal{E} + \left(\theta + \frac{3}{2}\Sigma\right)\mathcal{H}, \quad (66)$$

$$\begin{aligned} \dot{\mathcal{H}}_a - \frac{1}{2}\varepsilon_{ab}\hat{\mathcal{E}}^b &= \left(\frac{3}{4}\Sigma - \theta\right)\mathcal{H}_a - \frac{3}{2}\mathcal{E}\varepsilon_{ab}\mathcal{A}^b \\ &+ \frac{3}{4}\mathcal{E}\varepsilon_{ab}a^b - \frac{1}{2}\varepsilon_{bc}\delta^b\mathcal{E}_a^c \\ &+ \left(\frac{1}{4}\phi + \mathcal{A}\right)\varepsilon_{ab}\mathcal{E}^b - \frac{3}{4}\varepsilon_{ab}\delta^b\mathcal{E}, \end{aligned} \quad (67)$$

$$\begin{aligned} \dot{\mathcal{H}}_{\{ab\}} + \varepsilon_{c\{a}\hat{\mathcal{E}}_{b\}}^c &= +\frac{3}{2}\mathcal{E}\varepsilon_{c\{a}\zeta_{b\}}^c - \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{c\{a}\mathcal{E}_{b\}}^c \\ &- \left(\theta + \frac{3}{2}\Sigma\right)\mathcal{H}_{ab} + \varepsilon_{c\{a}\delta^c\mathcal{E}_{b\}}. \end{aligned} \quad (68)$$

In the above equation all the zeroth order quantities are background quantities. If the background is static with $\Theta = \Sigma = 0$ and the time derivative all the background quantities are zero, we can easily see that the time derivatives of the first order quantities at a given point is of the same order of smallness as themselves. Hence the first order quantities still remains “small” as the time evolves.

Similarly we can write the spatial propagation equation of all the first order quantities up to $\mathcal{O}(\epsilon)$. The propagation equations of ξ and $\zeta_{\{ab\}}$ are:

$$\dot{\xi} = -\phi\xi + \left(\frac{1}{3}\theta + \Sigma\right)\Omega + \frac{1}{2}\varepsilon_{ab}\delta^a a^b, \quad (69)$$

$$\dot{\zeta}_{\{ab\}} = -\phi\zeta_{ab} + \delta_{\{a}a_{b\}} + \left(\frac{1}{3}\theta + \Sigma\right)\Sigma_{ab} - \mathcal{E}_{ab}. \quad (70)$$

Shear divergence:

$$\begin{aligned} \dot{\Sigma}_a - \varepsilon_{ab}\hat{\Omega}^b &= \frac{1}{2}\delta_a\Sigma + \frac{2}{3}\delta_a\theta - \varepsilon_{ab}\delta^b\Omega - \frac{3}{2}\phi\Sigma_a - \frac{3}{2}\Sigma a_a \\ &+ \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{ab}\Omega^b - \delta^b\Sigma_{ab}, \end{aligned} \quad (71)$$

$$\begin{aligned} \dot{\Sigma}_{\{ab\}} &= \delta_{\{a}\Sigma_{b\}} - \varepsilon_{c\{a}\delta^c\Omega_{b\}} - \frac{1}{2}\phi\Sigma_{ab} \\ &+ \frac{3}{2}\Sigma\zeta_{ab} - \varepsilon_{c\{a}\mathcal{H}_{b\}}^c. \end{aligned} \quad (72)$$

Vorticity divergence equation:

$$\hat{\Omega} = -\delta_a \Omega^a + (\mathcal{A} - \phi) \Omega . \quad (73)$$

Electric Weyl Divergence:

$$\hat{\mathcal{E}}_a = \frac{1}{2} \delta_a \mathcal{E} - \delta^b \mathcal{E}_{ab} - \frac{3}{2} \mathcal{E} a_a - \frac{3}{2} \phi \mathcal{E}_a . \quad (74)$$

Magnetic Weyl divergence:

$$\hat{\mathcal{H}} = -\delta_a \mathcal{H}^a - \frac{3}{2} \phi \mathcal{H} - 3 \mathcal{E} \Omega , \quad (75)$$

$$\begin{aligned} \hat{\mathcal{H}}_a = & 12 \delta_a \mathcal{H} - \delta^b \mathcal{H}_{ab} - \frac{3}{2} \mathcal{E} \varepsilon_{ab} \Sigma^b + \frac{3}{2} \mathcal{E} \Omega_a \\ & + \frac{3}{2} \Sigma \varepsilon_{ab} \mathcal{E}^b - \frac{3}{2} \phi \mathcal{H}_a . \end{aligned} \quad (76)$$

Hence we see that if the background is spatially homogeneous with $\phi = \mathcal{A} = 0$ and the ‘hat’ derivative all the background quantities are zero, we can easily see that the ‘hat’ derivatives of the first order quantities at a given point are of the same order of smallness as themselves. Hence the first order quantities still remain “small” along the spatial direction. In both the cases of a static background and a spatially homogeneous background the resultant set of equations are the perturbed LRS-II equations, (that is equations (29)- (35) with $\mathcal{O}(\epsilon)$ terms added to each).

Again trying to solve the *Killing equation* (43) for a Killing vector of the form $\xi_a = \Psi u_a + \Phi e_a$, using equation (26),(25) and multiplying the Killing equation by $u^a u^b$, $u^a e^b$, $e^a e^b$, N^{ab} , $N_c^a u^b$, $N_c^a e^b$ and $N_c^a N_d^b$, we get the following differential equations and constraints:

$$\dot{\Psi} + \mathcal{A}\Phi = 0 , \quad (77)$$

$$\dot{\Psi} - \dot{\Phi} - \Psi \mathcal{A} + \Phi (\Sigma + \frac{1}{3} \Theta) = 0 , \quad (78)$$

$$\dot{\Phi} + \Psi (\frac{1}{3} \Theta + \Sigma) = 0 , \quad (79)$$

$$\Psi (\frac{2}{3} \Theta - \Sigma) + \Phi \phi = 0 , \quad (80)$$

$$-\delta_c \Psi + \Psi \mathcal{A}_c + \Phi (\varepsilon_{cd} \Omega^d + \alpha_c + \Sigma_c) = 0 , \quad (81)$$

$$\delta_c \Phi + \Phi a_c + 2 \Psi \Sigma_c = 0 , \quad (82)$$

$$\Psi \Sigma_{cd} + \Phi \zeta_{cd} = 0 . \quad (83)$$

Now we see that for both timelike ($\Phi = 0$) or spacelike ($\Psi = 0$) vectors, all the above equations are not completely solved in general unless the first order quantities appearing in the equations above are exactly equal to zero. This special case corresponds to static but distorted black holes in the presence of matter outside black hole, for example when an accretion disk occurs [12]. If the distribution of the matter outside black hole is axisymmetric, then the vacuum metric outside the matter is described by Weyl Solution. However as we proved that these first order quantities generically remain $\mathcal{O}(\epsilon)$ both in space and time, we can see that a timelike vector with ($\Theta = \Sigma = 0$) or a spacelike vector with ($\phi = \mathcal{A} = 0$) almost solves the Killing equations.

Therefore we can say:

For an almost spherically symmetric vacuum spacetime there always exists a vector in the local $[u, e]$ plane which almost solves the Killing equations. If this vector is timelike then the spacetime is locally almost static, and if the Killing vector is spacelike the spacetime is locally almost spatially homogeneous.

Also as we have seen that in this case the resultant set of equations are the perturbed LRS-II equations, (that is equations (29)- (35) with $\mathcal{O}(\epsilon)$ terms added to each), and the perturbations locally remain small both in space and time, a part of the maximally extended almost-Schwarzschild solution will then solve the field equations locally.

Thus we have proved the (local) **Almost Birkhoff Theorem**: *Any C^2 solution of Einstein’s equations in empty space, which is almost spherically symmetric in an open set \mathcal{S} , is locally almost equivalent to part of a maximally extended Schwarzschild solution in \mathcal{S} .*

Note that we do not consider perturbations across the horizon: our result holds for any open set \mathcal{S} that does not intersect the horizon in the background spacetime. The result almost certainly holds true across the horizon also, but that case needs separate consideration.

The above result can be immediately generalized in the presence of a cosmological constant. In that case an ‘almost’ spherically symmetric solution in an open set \mathcal{S} , is locally almost equivalent to part of a maximally extended Schwarzschild deSitter/anti-deSitter solution in \mathcal{S} . Also the result holds for an almost spherically symmetric electric charge distribution with no spin or magnetic dipole. In this case we have to use the energy momentum tensor of the electromagnetic field in vacuum with the magnetic part being equal to zero. Proceeding exactly in a similar manner one can then show that the solution of the perturbed field equations will be almost equivalent to a part of maximally extended Reissner-Nordström spacetime.

V. DISCUSSION

The rigidity of spherical vacuum solutions of the EFE, as enshrined in Birkhoff’s theorem, is maintained in the perturbed case: almost spherical symmetry implies almost static. This is an important reason for the stability of the solar system, and of black hole spacetimes.

Though there are many discussions in the literatures on the stability of Schwarzschild solution in General Relativity, for example [13]; most of them deal with a specific sector of the maximally extended Schwarzschild manifold, namely the static exterior part. In this paper we have established in a compact and completely different way, an aspect of the stability of the static sector of

the complete manifold: *as long as the solution remains almost spherically symmetric, it remains almost static*, with a similar result for the spatially homogeneous sector. Furthermore our result, being local, does not depend on specific boundary conditions used for solving the perturbation equations. Hence it does not depend on the global topology of the spacetime, and brings out covariantly the rigidity and uniqueness of the almost-spherical

vacuum solutions of Einstein's field equations.

This rigidity has interesting implications for the issue of how a universe made up of locally spherically symmetric objects imbedded in vacuum regions is able to expand, given that Birkhoff's theorem tells us the local spacetime domains have to be static. A two-mass exact solution illustrating this situation is given in [14]; the present paper suggests the results given there are stable.

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